Orbit equivalence, flow equivalence, and C*-algebras

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Overview

1. Cantor minimal systems
2. Symbolic dynamics I
3. Symbolic Dynamics II
4. Case study

Odometer

Let $q_{k+1}\colon \mathbb{Z}/2^{k+1}\mathbb{Z} \to \mathbb{Z}/2^k\mathbb{Z}$ be the canonical quotient map and consider the projective limit

$$X = \{(x_k)_k \in \prod_k \mathbb{Z}/2^k\mathbb{Z} : q_{k+1}(x_{k+1}) = x_k \forall k\}$$

with the transformation $T\colon X \to X$ given by $Tx_k = x_k + 1$.

Remark

Then X is the Cantor set, and T is a minimal homeomorphism.

Cantor minimal systems

Definition

A Cantor Minimal System (X, T) is a Cantor set X equipped with a minimal homeomorphism $T\colon X \to X$.

Corollary (GPS95)

Any (uniquely ergodic) Cantor minimal system is orbit equivalent to an odometer or a Denjoy system.

Cantor minimal systems

Theorem (GPS95)

For Cantor minimal systems (X_1, T_1) and (X_2, T_2) the following are equivalent:

1. *they are strong orbit equivalent;*
2. $K^0(X_1, T_1) \cong K^0(X_2, T_2)$ *as ordered groups with distinguished order units (ordered cohomology);*
3. $C(X_1) \rtimes_{T_1} \mathbb{Z} \cong C(X_2) \rtimes_{T_2} \mathbb{Z}$ *as C^*-algebras.*

Remark

The group $K^0(X, T) = C(X, \mathbb{Z})/\{f - f \circ T : f \in C(X, \mathbb{Z})\}$ is the ordered cohomology.

Sturmian subshifts

Let $\alpha \in (0,1) \setminus \mathbb{Q}$ and consider the rigid rotation $R_\alpha \colon [0,1) \longrightarrow [0,1)$ given by $R_\alpha(t) = t + \alpha \pmod 1$. Consider $I_\alpha \colon [0,1) \longrightarrow \{\mathtt{0}, \mathtt{1}\}$ given by

$$I_\alpha(t) = \begin{cases} \mathtt{0}, & t \in [0, 1-\alpha), \\ \mathtt{1}, & t \in [1-\alpha, 1). \end{cases}$$

The Sturmian shift is then

$$\bar{\mathsf{X}}_\alpha = \overline{\{(I_\alpha(R_\alpha^i(t)))_{i\in\mathbb{Z}} : t \in [0,1)\}} \subset \{\mathtt{0},\mathtt{1}\}^{\mathbb{Z}}$$

with the shift transformation $\sigma_\alpha \colon \bar{X}_\alpha \to \bar{X}_\alpha$.

Sturmian subshifts

Note

The Sturmian system $(\bar{X}_\alpha, \bar{\sigma}_\alpha)$ is a Cantor minimal system (a Denjoy system).

- $C(\bar{X}_\alpha) \rtimes \mathbb{Z}$ is understood ($A\mathbb{T}$-algebra, real rank zero, stable rank 1, $\ldots$);
- $K^0(\bar{X}_\alpha, \bar{\sigma}_\alpha) = \mathbb{Z} + \alpha\mathbb{Z} \subset \mathbb{R}$

Overview

Shifts of finite type

The edge shift of directed graph E with adjacency matrix A is the compact space

$$\bar{\mathsf{X}}_A = \{x = (x_n)_{n\in\mathbb{Z}} \in (E^1)^{\mathbb{Z}} : r(x_n) = s(x_{n+1}), n \in \mathbb{Z}\}$$

equipped with the shift $\sigma_A(x)_n = x_{n+1}$ for $x \in \bar{\mathsf{X}}_A$ and $n \in \mathbb{Z}$.

Example

The graph defines the full 2-shift with path space $\bar{\mathsf{X}}_{[2]} = \{0,1\}^{\mathbb{Z}}$.

Cuntz–Krieger algebras

Theorem (Cuntz–Krieger+)

To every (irreducible and nonpermutation) $\mathbb{N}$-matrix A, there exists a universal unital C^*-algebra

$$\mathcal{O}_A = \overline{span}\{s_\alpha {s_\beta}^* : \alpha, \beta \text{ finite paths}\}$$

generated by $|A|$ partial isometries s_i. There is a canonical gauge action $\gamma^A \colon \mathbb{T} \curvearrowright \mathcal{O}_A$ given by

$$\gamma_z^A(s_i) = z s_i,$$

and a (maximal) diagonal subalgebra

$$\mathcal{D}_A = \overline{span}\{s_\alpha s_\alpha^* : \alpha \text{ a finite path}\}$$

Cuntz–Krieger algebras

Observations:

A	$\mathcal{O}_A$
Irreducibility	Simplicity
Irreducible components	Ideal structure
Path space: X_A	$\mathcal{D}_A = C(\mathsf{X}_A)$ (Diagonal)
Time stretching	Gauge action γ^A
Perron eigenvalue (entropy)	KMS-structure

Shifts of finite type

Definition

A pair of shifts $\bar{\mathsf{X}}_A$ and $\bar{\mathsf{X}}_B$ are conjugate if there is a homeomorphism $h\colon \bar{\mathsf{X}}_A \to \bar{\mathsf{X}}_B$ satisfying $h \circ \sigma_A = \sigma_B \circ h$.

Example

Are the systems given by the matrices

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 12 \\ 1 & 1 \end{bmatrix}$$

conjugate?

Flow Equivalence

Flow equivalence is a coarse (but important!) relation in symbolic dynamics.

Definition

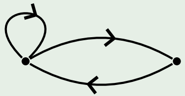
From (X, T), we construct

- $X \times \mathbb{R}$ and $(\sigma(x), t) \sim (x, t+1)$;
- $X \times \mathbb{R}/\sim$ is the mapping torus with induced $\mathbb{R}$-flow.

Flow Equivalence

Example

The graphs

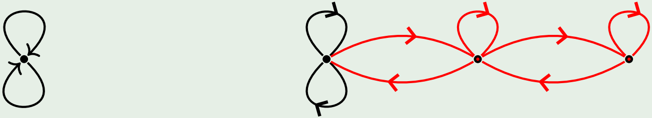


are Flow Equivalent (time delay).

Flow Equivalence

Example (Cuntz splice)

The graphs



are not flow equivalent: different sign of determinant $I - A$.

Flow equivalence

For irreducible shifts of finite type, the following are equivalent:

1. $(\bar{\mathsf{X}}_A, \sigma_A)$ and $(\bar{\mathsf{X}}_B, \sigma_B)$ are flow equivalent;
2. $K^0(\bar{X}_A, \sigma_A) \cong K^0(\bar{\mathsf{X}}_B, \sigma_B)$ as ordered groups (no unit);
3. $K_0(\mathcal{O}_A) \cong K_0(\mathcal{O}_B)$ and $\det(I - A) = \det(I - B)$;
4. $[\mathcal{O}_A \otimes \mathbb{K}, \mathcal{D}_A \otimes c_0] \cong [\mathcal{O}_B \otimes \mathbb{K}, \mathcal{D}_B \otimes c_0]$.

(Cuntz–Krieger, Boyle–Handelman, Bowen–Franks, Matsumoto–Matui)

Problem.

- Classify shifts of finite type up to orbit equivalence.
- What does ordered cohomology with unit reflect?

Overview

General subshifts

A general subshift is a closed subset $X \subset \{1, \cdots, N\}^{\mathbb{Z}}$ which is shift-invariant.

Problem.

Classify subshifts up to flow equivalence.

General subshifts

Start with a subshift (X, σ_X):

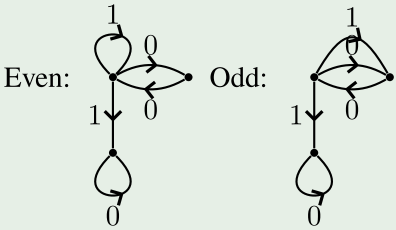
1. construct the cover $(\tilde{X}, \sigma_{\tilde{X}})$;
2. construct the Deaconu–Renault groupoid G_X with unit space $\tilde{X}$;
3. let $\mathcal{O}_X = C^*(G_X)$ with $C(X) \subset \mathcal{O}_X$ and gauge action $\gamma^X : \mathbb{T} \curvearrowright \mathcal{O}_X$.

Problem.

How does $\mathcal{O}_X$ remember the underlying dynamics?

Sofic subshifts

Example



The map defined by sending $1 \longmapsto 10$ defines a flow equivalence.

Sofic subshifts

Theorem (B-Carlsen2020)

Sofic subshifts $(\bar{X}, \sigma_{\bar{X}})$ *and* $(\bar{Y}, \sigma_{\bar{Y}})$ *(with no periodic points isolated in past equivalence) are flow equivalent if and only if*

$$[\mathcal{O}_X \otimes \mathbb{K}, C(X) \otimes c_0] \cong [\mathcal{O}_Y \otimes \mathbb{K}, C(Y) \otimes c_0].$$

Is this a general phenomenon?

C*-results

Let $(\bar{X}, \sigma_{\bar{X}})$ and $(\bar{Y}, \sigma_{\bar{Y}})$ be subshifts.

Theorem (B-Carlsen2020)

1. *$(\bar{X}, \sigma_{\bar{X}})$ and $(\bar{Y}, \sigma_{\bar{Y}})$ are conjugate if and only if*

$$[\mathcal{O}_X \otimes \mathbb{K}, C(X) \otimes c_0, \gamma^X \otimes \mathrm{id}] \cong [\mathcal{O}_Y \otimes \mathbb{K}, C(Y) \otimes c_0, \gamma^Y \otimes \mathrm{id}];$$

2. *If $(\bar{X}, \sigma_{\bar{X}})$ and $(\bar{Y}, \sigma_{\bar{Y}})$ are flow equivalent, then*

$$[\mathcal{O}_X \otimes \mathbb{K}, C(X) \otimes c_0] \cong [\mathcal{O}_Y \otimes \mathbb{K}, C(Y) \otimes c_0].$$

We need to investigate invariants of $\mathcal{O}_X \otimes \mathbb{K}$ with $C(X) \otimes c_0$.

Overview

Case study: Sturmian subshifts

Let $\alpha \in (0,1) \setminus \mathbb{Q}$ and consider the rigid rotation $R_\alpha \colon [0,1) \longrightarrow [0,1)$ given by $R_\alpha(t) = t + \alpha \pmod 1$.
Consider $I_\alpha \colon [0,1) \longrightarrow \{\mathtt{0}, \mathtt{1}\}$ given by

$$I_\alpha(t) = \begin{cases} \mathtt{0}, & t \in [0, 1-\alpha), \\ \mathtt{1}, & t \in [1-\alpha, 1). \end{cases}$$

The *Sturmian subshift* is then

$$\mathsf{X}_\alpha = \overline{\{(I_\alpha(R_\alpha^i(t)))_{i\in\mathbb{Z}} : t \in [0,1)\}} \subset \{\mathtt{0}, \mathtt{1}\}^{\mathbb{Z}}.$$

Case study: Sturmian subshifts

The shift operation $\sigma_\alpha : \bar{\mathsf{X}}_\alpha \to \bar{\mathsf{X}}_\alpha$ defines a Cantor minimal system.
Two distinct C^*-algebras:

1. the crossed product $C(\bar{\mathsf{X}}_\alpha) \rtimes \mathbb{Z}$;
2. the shift space C^*-algebra $\mathcal{O}_\alpha$.

Lemma (Carlsen)

There is a short exact sequence

$$0 \to \mathbb{K} \to \mathcal{O}_\alpha \to C(\bar{\mathsf{X}}_\alpha) \rtimes \mathbb{Z} \to 0$$

Case study: Sturmian subshifts

Theorem

Let $\alpha, \beta \in (0,1) \setminus \mathbb{Q}$. Then conjugacy is equivalent to:

- $\mathcal{O}_\alpha \cong \mathcal{O}_\beta$;
- $\alpha = \beta, 1-\beta$;
- $\mathbb{Z} + \alpha\mathbb{Z} \cong \mathbb{Z} + \beta\mathbb{Z}$ *with order and with unit.*

Theorem

Let $\alpha, \beta \in (0,1) \setminus \mathbb{Q}$. Then flow equivalence is equivalent to:

- $\mathcal{O}_\alpha \otimes \mathbb{K} \cong \mathcal{O}_\beta \otimes \mathbb{K}$;
- $\alpha \sim \beta$ *(equivalence of irrationals)*;
- $\mathbb{Z} + \alpha\mathbb{Z} \cong \mathbb{Z} + \beta\mathbb{Z}$ *with order and without unit.*

Case study: Sturmian subshifts

Let $(X_\alpha, \sigma_\alpha)$ be Sturmian.

Theorem (B)

- *The cover is* $\tilde{X}_\alpha = \bar{X}_\alpha \cup \{$*countable discrete orbit*$\}$;
- *The* C^**-algebra* $\mathcal{O}_\alpha$ *is infinite (not properly infinite), has real rank zero and stable rank 2, and nuclear dimension one.*

- Operator algebraic approach to the flow equivalence problem;
- Operator algebraic approach to the conjugacy problem;
- Emphasis on one-sided versions;
- In depth study of Sturmian systems (nuclear dimension);
- Noncommutative dynamical systems and Cuntz–Pimsner algebras. (in progress)