OPEN PROBLEMS FROM THE GLASGOW WORKSHOP, AUGUST 2022

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Contents

Introduction

The *Glasgow Late August Symbolic dynamics, Groups, and Operators Workshop* was a workshop held at the University of Glasgow in August 2022 organised by Kevin Aguyar Brix, Chris Bruce, Se-Jin (Sam) Kim, Xin Li, Alistair Miller, and Owen Tanner. The meeting aimed to include especially young mathematicians into the fascinating interactions between group theory, dynamical systems, and operator algebras. The workshop spanned a week (Monday to Friday) and Thursday afternoon had a session dedicated to the discussion and introduction of open problems. This document serves as a rough summary of that discussion.

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1. Groupoids, homology, and K-theory

In [Ma12, Ma16], Matui observed that for groupoids constructed e.g. from minimal integer actions and (products of) one-sided shifts of finite type there is an isomorphism

$$K_i(C_r^*(G)) \cong \bigoplus_{n\geqslant 0} H_{2n+i}(G),$$

for $i = 0, 1$. He went on to conjecture that this could be the case for all minimal, effective, and ample groupoids G with compact unit space (Matui's HK conjecture). The conjecture was verified for many classes of groupoids but Scarparo [Sc19] gave the first counterexample (odometers have torsion in isotropy). Deeley [De22] later found a *principal* counterexample and noted that these examples satify a *rational HK conjecture*. This leads to the following adjusted conjecture, cf. [BDGW].

Problem 1.1. Given an ample groupoid G with torsion-free isotropy and satisfying the Baum–Connes conjecture, are there isomorphisms

$$K_i(C_r^*(G)) \otimes \mathbb{Q} \cong \bigoplus_{n\geqslant 0} H_{2n+i}(G,\mathbb{Q}),$$

for $i = 0, 1$?

2. Equivariant and diagonal isomorphism of graph C*-algebras

A square matrix A with entries in $\mathbb{N}$ may be interpreted as the adjacency matrix of finite directed graph, and the Cuntz–Krieger algebra $\mathcal{O}_A$ associated to this matrix is canonically isomorphic to the graph C*-algebra. This C*-algebra comes equipped with a commutative diagonal subalgebra, denoted $\mathcal{D}_A$, as well as a canonical gauge action γ^A.

Problem 2.1. When does an equivariant isomorphism of stable Cuntz–Krieger algebras imply a diagonal-preserving isomorphism of Cuntz–Krieger algebras? More precisely, when does the existence of a *-isomorphism $(\phi\colon \mathcal{O}_A\otimes\mathbb{K}) \to \mathcal{O}_B\otimes\mathbb{K}$ satisfying $\gamma^B\otimes\mathrm{id}\circ\phi = \phi\circ(\gamma^A\otimes\mathrm{id})$ imply the existence of a *-isomorphism $\phi'\colon \mathcal{O}_A\otimes\mathbb{K}\to\mathcal{O}_B\otimes\mathbb{K}$ satisfying $\phi'(\mathcal{D}_A\otimes c_0) = \mathcal{D}_B\otimes c_0$?

When the matrix is primitive (i.e. there exists a positive integer N such that A^N contains only positive integers), then the dynamical relation *shift equivalence* of the two-sided edge shift spaces associated to the graphs is equivalent to a stable and equivariant *-isomorphisms of the Cuntz–Krieger algebras. Similarly, *flow equivalence* of same edge shift spaces is characterised by diagonal-preserving stable isomorphism, and it is known that in this case shift equivalence implies flow equivalence. Note that by the counterexamples to the Williams conjecture by Kim and Roush [KR99], we cannot in general expect ϕ' above to be the same as ϕ.

3. Shift equivalent C*-correspondences

In [KK14] (see also [KK22]), Kakariadis and Katsoulis introduce the notion of *shift equivalence* for regular C*-correspondences: a pair of C*-correspondences E and F over C*-algebras A and B, respectively, are *shift equivalent* if there are C*-correspondences R and S and a positive integer m (the lag) such that

$$E^{\otimes m} \cong R\otimes_A S, \quad F^{\otimes m}\cong S\otimes_B R, \quad S\otimes_A E\cong F\otimes_B S, \quad E\otimes_A R\cong R\otimes_B F. \qquad (3.1)$$

Let X be a regular C*-correspondence over a C*-algebra A and let $\mathcal{O}_X$ be its Cuntz–Pimsner algebra. The *Pimsner dilation* of X is given as follows. First, let A_∞ be the fixed-point algebra $\mathcal{O}_X^\gamma$ of the Cuntz–Pimsner algebra with respect to its canonical gauge action γ. The module $X\otimes_A A_\infty$ is then a correspondence over A_∞, and the Cuntz–Pimsner algebras $\mathcal{O}_{X_\infty}$ and $\mathcal{O}_X$ are *-isomorphic (see [KK14, Section 3] for details and why this is well defined). In fact, when X is full and nondegenerate, the Pimsner dilation X_∞ is an imprimitivity bimodule.

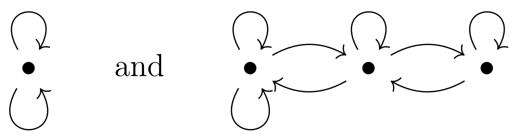
The question below was postulated in [KK14, Theorem 5.8]; however, the recent preprint [CDE] points out a subtle gap that renders the proof invalid.

Problem 3.1. Let E and F be regular C*-correspondences over C*-algebras A and B, respectively. If E and F are shift equivalent, does it follow that their Pimsner dilations E_∞ and F_∞ are shift equivalent?

A positive answer to this question would entail that a shift of finite type is completely characterised up to shift equivalence by the stabilised gauge action on its Cuntz–Krieger algebra.

4. The Cuntz splice

Consider the graphs



The first graph C*-algebra is Cuntz' algebra $\mathcal{O}_2$ while the other one is historically called $\mathcal{O}_{2-}$. By using deep classification results Rørdam [Rø95] showed that $\mathcal{O}_2$ and $\mathcal{O}_{2-}$ are in fact *-isomorphic. It is however easy to show that the shifts of finite type they determine are not flow equivalent, so this isomorphism cannot be neither diagonal-preserving nor gauge-equivariant. To this day, the corresponding algebraic question concerning the Leavitt path algebras remains open:

Problem 4.1. Are the Leavitt path algebras L_2 and L_{2-} (over any field) isomorphic?

For Leavitt path algebras over the integers such an isomorphism does not exist [JS16]. According to [RT13], this is the last missing step in the classification of unital Leavitt path algebras.

The fact that the shifts of finite type are not flow equivalent also implies that the topological full groups (of the associated boundary path groupoids) are not isomorphic. For the first graph, the group is Thompson's group V, and we call the other group V_{2-}.

Problem 4.2. Is there a group-theoretic way to distinguish V and V_{2-}?

5. Amenability of self similar groups

Let X be a finite set (the alphabet) and let X^* be the collection of all finite words in the alphabet. A *self-similar group* is then group G with a faithful action on X^* such that for every $g \in G$ and $x \in X$ there exist $h \in G$ and $y \in X$ such that

$$g(xw) = yh(w)$$

for all $w \in X^*$. In fact, for $g \in G$ and $v \in X^*$ there is a unique element $g|_v \in G$ (the restriction of g at v) such that $g(vw) = g(v)g|_v(w)$ A self-similar group G is *contracting* if it admits a finite subset $\mathcal{N}$ (the nucleus) that contains the restriction of any group element at a sufficiently long finite word.

Examples of self-similar groups arise from *iterated monodromy groups* of expanding self-coverings of compact path-connected metric spaces. In particular, the iterated monodromy group of the complex polynomial $z \mapsto z^2 - 1$ was the first example of an amenable group which cannot be constructed from groups with subexponential growth using basic group operations [Ne05, Theorem 6.12.1].

Problem 5.1. Are all faithful contracting self-similar groups amenable?

This problem appears in [Ne05, Section 6.12] where many examples are also given.

6. Topological full groups

Let X be the Cantor space and let $G \curvearrowright X$ be an action on X of a countable discrete group G. The *topological full group* of the action is the subgroup of homeomorphisms on X that locally look like the action. (This notion can also be defined for ample groupoids using compact open bisections). On the other hand, it is an important open problem in group theory whether every finitely presented simple group is two-generated.

In [BEH20], Bleak, Elliott, and Hyde introduce the notion of a subgroup of the homeomorphisms on the Cantor space to be *vigorous* (any countably infinite group can be realised as homeomorphisms on the Cantor space). They show that vigorous, simple, and finitely generated groups are in fact two-generated. Examples include Thompson's group V and all groups it embeds into such as Röver–Nekrashevych groups, Higman–Thompson groups, and Brin–Thompson groups. In many interesting cases, the topological full groups are known to be simple and finitely generated, so this leads to the following question.

Problem 6.1. When are topological full groups vigorous?

For the case when the topological full groups are finitely presented we may also ask:

Problem 6.2. When is the isomorphism problem for topological full groups decidable?

This is particular interesting when these groups come from expanding self-coverings of compact path-connected metric spaces (as in Section 5) since the topological full groups completely characterise the expanding maps up to conjugacy (they can be built using the iterated monodromy groups). Similarly, for one-sided shifts of finite type, the topological full groups completely characterise the dynamical system up to continuous orbit equivalence, and this is in fact known to be decidable [BS20].

7. Comparison of group actions

Let $G \curvearrowright X$ be an action of a discrete group G on a compact metrisable space X. For a pair of subsets A and B of X we write $A \preceq B$ if for every closed subset C of A there exists a finite collection $\mathcal{U}$ of open sets that cover C and elements $s_U \in G$ for each $U \in \mathcal{U}$ such that $s_U U$ are pairwise disjoint in B. The action $G \curvearrowright X$ is then said to have *comparison* if $A \preceq B$ for all nonempty open subsets A and B of X whenever $\mu(A) < \mu(B)$ for all G-invariant measures on X. Via Kerr's notion of almost finiteness group actions of amenable groups [Ke20] (see also [KS20]), comparison plays a key role in establishing $\mathcal{Z}$-stability of the crossed product $G \ltimes C(X)$.

Problem 7.1. Does every free and minimal action $G \curvearrowright X$ of a discrete countable group G on a compact metric space X have comparison? What if G is amenable?

When the group is finitely generated with polynomial growth then this problem has a positive answer, cf. [Na22, Theorem A].

8. Ultrafilters, ultraproducts, and elementary equivalence

A famous paper of Ge and Hadwin [GH01] showed that the Continuum Hypothesis is equivalent to the C^*-ultraproducts of separable C^*-algebras being isomorphic for any ultrafilters on the natural numbers, and that the tracial ultrapowers of the hyperfinite II_1 factor $\mathcal{R}$ are isomorphic for any free ultrafilter. Moreover the Continuum Hypothesis implies this isomorphism for *tracial* ultrapowers of finite von Neumann algebras with separable predual. Farah, Hart, and Sherman [FHS13, Proposition 3.3] then showed that if the Continuum Hypothesis fails, then the matrix algebras $M_n(\mathbb{C})$ for $n \in \mathbb{N}$ have non-isomorphic tracial ultraproducts. The complementary problem is then posed as [FHS14, Question 6.6], see also [Je22, Question 5.6]. Two models are *elementary equivalent* if their theories are equal, cf. [FHS14, Section 2.5].

Problem 8.1. Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on the natural numbers. Are the (tracial) ultraproducts $\prod_{n\to\mathcal{U}} M_n(\mathbb{C})$ and $\prod_{n\to\mathcal{V}} M_n(\mathbb{C})$ necessarily elementary equivalent? Assuming the Continuum Hypothesis, are they isomorphic?

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